

# The Weibull Birnbaum-Saunders Distribution: Properties and Applications

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## Abstract

This paper introduces a new four-parameter lifetime model called the Weibull Birnbaum-Saunders distribution. This new distribution represents a more flexible model for the lifetime data. Its failure rate function can be increasing, decreasing, upside-down bathtub shaped, bathtub-shaped or modified bathtub shaped depending on its parameters. Some structural properties of the proposed model are investigated including expansions for the cumulative and density functions, moments, generating function, mean deviations, order statistics and reliability. The maximum likelihood estimation method is used to estimate the model parameters and the observed information matrix is determined. The flexibility of the new model is shown by means of two real data sets.

**Keywords:** Birnbaum-Saunders distribution, Weibull-G class, moment, order statistic, maximum likelihood estimation, observed information matrix.

## 1 Introduction

The two-parameter Birnbaum-Saunders (BS) distribution, which was introduced by Birnbaum and Saunders [3], is a very important lifetime model and is widely used in reliability studies. This distribution, also known as the fatigue life distribution, was originally derived from a model that shows the total time that passes until that some type of cumulative damage, produced by the development and growth of a dominant crack, surpasses a threshold

value and causes the material specimen to fail. Desmond [10] provided a more general derivation based on a biological model and strengthened the physical justification for the use of this distribution.

A random variable  $T$  having the BS distribution with shape parameter  $\alpha > 0$  and scale parameter  $\beta > 0$ , denoted by  $T \sim \text{BS}(\alpha, \beta)$ , is defined by

$$T = \beta \left[ \frac{\alpha Z}{2} + \sqrt{\left( \frac{\alpha Z}{2} \right)^2 + 1} \right]^2,$$

where  $Z$  is a standard normal random variable. The cumulative distribution function (cdf) of  $T$  is

$$G(t) = \Phi(v), \quad t > 0, \quad (1)$$

where  $\Phi(\cdot)$  is the standard normal distribution function,  $v = \alpha^{-1}\rho(t/\beta)$  and  $\rho(z) = z^{1/2} - z^{-1/2}$ . The probability density function (pdf) corresponding to (1) is given by

$$g(t) = \kappa(\alpha, \beta) t^{-3/2} (t + \beta) \exp \left\{ -\frac{\tau(t/\beta)}{2\alpha^2} \right\}, \quad t > 0, \quad (2)$$

where  $\kappa(\alpha, \beta) = \exp(\alpha^{-2}) / (2\alpha\sqrt{2\pi\beta})$  and  $\tau(z) = z + z^{-1}$ . The fractional moments of  $T$  are given by (see [20])

$$E(T^k) = \beta^k I(\alpha, \beta),$$

where

$$I(\alpha, \beta) = \frac{K_{k+1/2}(\alpha^{-2}) + K_{k-1/2}(\alpha^{-2})}{2K_{1/2}(\alpha^{-2})}, \quad (3)$$

and the function  $K_\nu(z)$  denotes the modified Bessel function of the third kind with  $\nu$  representing its order and  $z$  the argument. The parameter  $\beta$  is the median of the BS distribution, because  $G(\beta) = \Phi(0) = 1/2$ .

Since the BS distribution was proposed, it has received much attention in the literature. This attention for the BS distribution is due to its many attractive properties and its relation to the normal distribution. For more details on the BS distribution, we refer to [15] and the references therein. The BS distribution has been used in several research areas such as engineering, environmental sciences, finance, and wind energy. However, it allows for upside-down hazard rates only (see [14]), hence cannot provide reasonable fits for modeling phenomenon with decreasing, increasing, modified bathtub shaped and bathtub-shaped failure rates which are common in reliability studies.

For this reason, several generalizations and extensions of the BS distribution have been proposed in the literature. For example, Cordeiro and Lemonte [6], using the beta-G class [11], proposed an extension of BS distribution named as the beta BS distribution. Saulo et al. [22], based on the work of Cordeiro and de Castro [5], defined the Kumaraswamy BS distribution. Lemonte [16], based on the scheme introduced by Marshall and Olkin [17], defined the Marshall–Olkin extended BS distribution. Cordeiro et al. [7] adopted the McDonald-G class [2] to define the McDonald BS distribution. Cordeiro et al. [8] used the generator approach of Zografos and Balakrishnan [28] to introduce the gamma Birnbaum-Saunders distribution. In this paper, a new four-parameter extension for the BS distribution is proposed.

Recently, Bourguignon et al. [4] proposed an interesting method of adding a new parameter to an existing  $G$  distribution. The resulting distribution, known as the Weibull-G distribution, gives more flexibility to model various types of data. Let  $G(t, \theta)$  be a continuous baseline distribution with density  $g$  depends on a parameter vector  $\theta$  and the Weibull cdf  $F_W(w) = 1 - e^{-aw^b}$  (for  $w > 0$ ) with positive parameters  $a$  and  $b$ . Bourguignon et al. [4] replaced the argument  $w$  by  $G(w, \xi) / \overline{G}(w, \theta)$  where  $\overline{G}(w, \xi) = 1 - G(w, \theta)$ , and defined the cdf of their class by

$$F(t; a, b, \theta) = ab \int_0^{\left[\frac{G(t, \theta)}{\overline{G}(t, \theta)}\right]} w^{b-1} e^{-aw^b} dw = 1 - \exp \left( -a \left[ \frac{G(t, \theta)}{\overline{G}(t, \theta)} \right]^b \right). \quad (4)$$

Then, the Weibull-G density function is given by

$$f(t; a, b, \theta) = abg(t, \theta) \left[ \frac{G(t, \theta)^{b-1}}{\overline{G}(t, \theta)^{b+1}} \right] \exp \left( -a \left[ \frac{G(t, \theta)}{\overline{G}(t, \theta)} \right]^b \right). \quad (5)$$

Some generalized distributions have been proposed under this methodology. Tahir et al. [23, 24, 25] defined the Weibull-Pareto, Weibull-Lomax and Weibull-Dagum distributions by taking  $G(t, \theta)$  to be the cdf of the Pareto, Lomax and Dagum distributions, respectively. More recently, Afify et al. [1] defined and studied the Weibull Fréchet distribution. In a similar way, we propose a new extension for the BS distribution called the Weibull BS (WBS) distribution, which has been applied to the modeling of fatigue failure times and reliability studies.

The rest of the paper is organized as follows. In Section 2, we introduce the WBS distribution and plot its density and failure rate functions. In Section 3, we provide a mixture representation for its density and cumulative distributions. Structural properties such as the ordinary moments, generating

function, quantile function and simulation, mean deviations, the density of the order statistics and the reliability are derived in Section 4. In Section 5, we discuss maximum likelihood estimation of the WBS parameters and derive the observed information matrix. Two applications are presented in Section 6 to show the potentiality of the new distribution. Some concluding remarks are given in Section 7.

## 2 The WBS distribution

Substituting (1) in (4), the cdf of the WBS distribution can be written as

$$F(t) = 1 - \exp \left( -a \left[ \frac{\Phi(v)}{1 - \Phi(v)} \right]^b \right). \quad (6)$$

The pdf corresponding to (6) is given by

$$f(t) = ab\kappa(\alpha, \beta) t^{-3/2} (t + \beta) \exp \left( -\frac{\tau(t/\beta)}{2\alpha^2} \right) \times \left[ \frac{\Phi(v)^{b-1}}{\{1 - \Phi(v)\}^{b+1}} \right] \exp \left( -a \left[ \frac{\Phi(v)}{1 - \Phi(v)} \right]^b \right), \quad (7)$$

where  $\beta$  is a scale parameter and  $\alpha$ ,  $a$  and  $b$  are positive shape parameters. It is clear that the BS distribution is not a special case of WBS distribution. If a random variable  $T$  follows a WBS distribution with parameters  $\alpha$ ,  $\beta$ ,  $a$  and  $b$  will be denoted by  $T \sim \text{WBS}(\alpha, \beta, a, b)$ . The reliability and the failure rate function of  $T$  are, respectively, given by

$$R(t) = \exp \left( -a \left[ \frac{\Phi(v)}{1 - \Phi(v)} \right]^b \right),$$

and

$$h(t) = ab\kappa(\alpha, \beta) t^{-3/2} (t + \beta) \left[ \frac{\Phi^{b-1}(v)}{\{1 - \Phi(v)\}^{b+1}} \right] \exp \left\{ -\frac{\tau(t/\beta)}{2\alpha^2} \right\}.$$

Plots of pdf and failure rate function of the WBS distribution for selected values of the parameters are given in Figure 1 and Figure 2, respectively. Figure 1 indicates that the WBS pdf can take various shapes such as symmetric, right-skewed and left-skewed depending on the parameter values. Figure 2 shows that the failure rate function of the WBS distribution can be increasing, decreasing, upside-down bathtub (unimodal) shaped, bathtub-shaped or

modified bathtub shaped (unimodal shape followed by increasing) depending on the parameter values. So, the WBS distribution is quite flexible and can be used effectively in analyzing survival data.

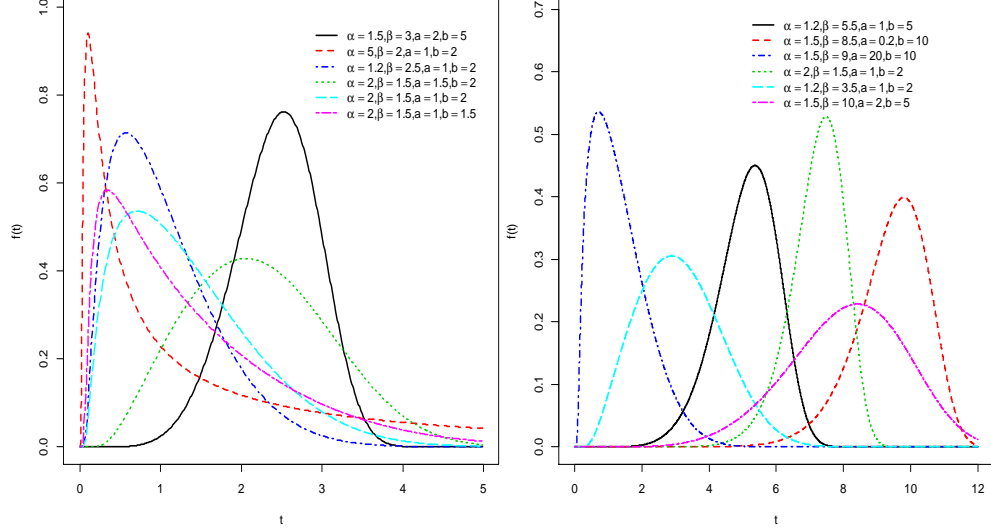


Figure 1. Plots of the WBS pdf for some values of the parameters.

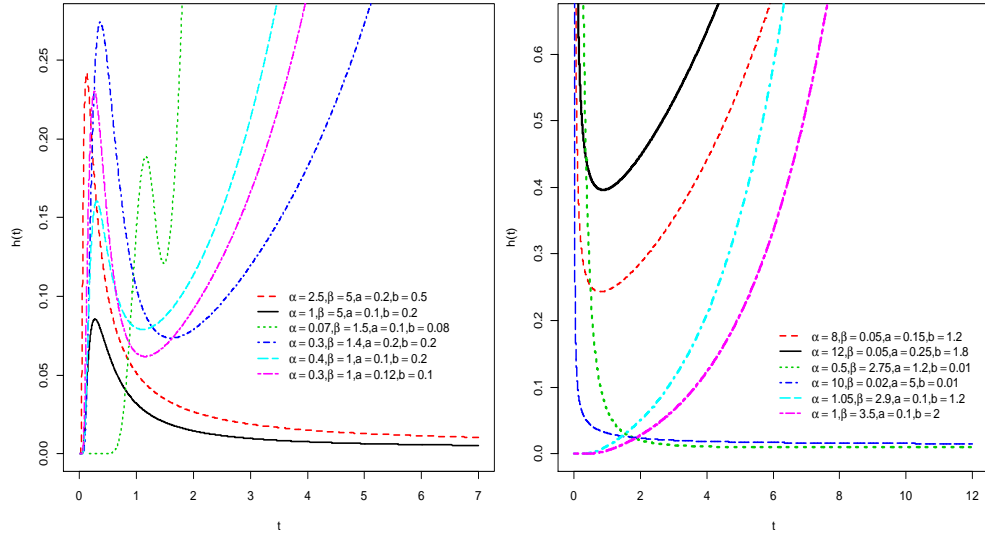


Figure 2. Plots of the WBS failure rate function for some values of the parameters.

### 3 Mixture representation

In this section, we derive expansions for the pdf and cdf of the WBS distribution. The pdf and cdf of the WBS distribution can be written as a linear combination of the pdf and cdf of exponentiated BS (EBS) distribution, respectively. A random variable  $X$  having the EBS distribution with parameters  $\alpha, \beta$  and  $a > 0$ , denoted by  $X \sim \text{EBS}(\alpha, \beta, a)$ , if its cdf and pdf are given by  $H(x) = \Phi^a(v)$  and  $h(x) = ag(x)\Phi^{a-1}(v)$ , respectively, where  $v$  is defined in (1) and  $g$  is given in (2). The properties of exponentiated distributions have been studied by several authors. For example, Mudholkar and Srivastava [19] studied the exponentiated Weibull distribution, Gupta and Kundu [13] studied the exponentiated exponential distribution and Sarhan and Apaloo [21] proposed the exponentiated modified Weibull extension distribution.

The pdf of the WBS distribution (7) can be written as

$$f(t) = abg(t) \frac{\Phi^{b-1}(v)}{[1 - \Phi(v)]^{b+1}} \exp\left(-a \left[\frac{\Phi(v)}{1 - \Phi(v)}\right]^b\right). \quad (8)$$

Using the series expansion for the exponential function, we obtain

$$\exp\left(-a \left[\frac{\Phi(v)}{1 - \Phi(v)}\right]^b\right) = \sum_{k=0}^{\infty} \frac{(-1)^k a^k}{k!} \frac{\Phi^{bk}(v)}{[1 - \Phi(v)]^{bk}}. \quad (9)$$

Substituting (9) in (8), we get

$$f(t) = abg(t) \sum_{k=0}^{\infty} \frac{(-1)^k a^k}{k!} \Phi^{bk+b-1}(v) [1 - \Phi(v)]^{-(bk+b+1)}.$$

Since  $0 < \Phi(v) < 1$ , for  $t > 0$  and  $(bk + b + 1) > 0$ , then by using the binomial series expansion  $[1 - \Phi(v)]^{-(bk+b+1)}$  given by

$$[1 - \Phi(v)]^{-(bk+b+1)} = \sum_{j=0}^{\infty} \frac{\Gamma(bk + b + 1 + j)}{j! \Gamma(bk + b + 1)} \Phi^j(v),$$

where  $\Gamma(\cdot)$  is the complete gamma function, we obtain

$$f(t) = \sum_{k,j=0}^{\infty} w_{k,j} h_{bk+b+j}(t), \quad (10)$$

where

$$w_{k,j} = \frac{(-1)^k b a^{k+1} \Gamma(bk + b + 1 + j)}{k! j! (bk + b + j) \Gamma(bk + b + 1)},$$

and  $h_{bk+b+j}(t)$  denotes the  $\text{EBS}(\alpha, \beta, bk + b + j)$  density function. By integrating (10), we get

$$F(t) = \sum_{k,j=0}^{\infty} w_{k,j} \Phi^{bk+b+j}(v). \quad (11)$$

It is easy to see that  $\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} w_{k,j} = 1$ . Equation (10) means that the pdf of the WBS distribution is a double linear mixture of the pdf of EBS distribution. Based on this equation, several structural properties of the WBS distribution can be obtained by knowing those of the EBS distribution. For example, the ordinary, inverse and factorial moments, generating function and characteristic function of the WBS distribution can be obtained directly from the EBS distribution.

If  $b$  is a positive real non-integer, we can expand  $\Phi^{bk+b+j}(v)$  as

$$\Phi^{bk+b+j}(v) = \sum_{r=0}^{\infty} s_r(bk + b + j) \Phi^r(v), \quad (12)$$

where

$$s_r(m) = \sum_{l=r}^{\infty} (-1)^{l+r} \binom{m}{l} \binom{l}{r}.$$

Thus, from equations (2), (10) and (12), we get

$$f(t) = g(t) \sum_{r=0}^{\infty} d_r \Phi^r(v), \quad (13)$$

where

$$d_r = \sum_{k,j=0}^{\infty} w_{k,j} s_r(bk + b + j).$$

## 4 Some structural properties

In this section, we give some mathematical properties of the WBS distribution.

### 4.1 Moments

In this subsection, we derive the expression for  $s$ th order moment of WBS distribution. The moments of some orders will help in determining the expected life time of a device and also the dispersion, skewness and kurtosis in a

given set of observations arising in reliability applications. The  $s$ th moment of the WBS random variable  $T$  can be derived from the probability weighted moments of the BS distribution. The probability weighted moments of the BS distribution are formally defined, for  $p$  and  $r$  non-negative integers, by

$$\tau_{p,r} = \int_0^\infty t^p g(t) \Phi^r(v) dt. \quad (14)$$

There are many softwares such as MAPLE, MATLAB and R that can be used to compute the integral (14) numerically. From [6], we have an alternative representation to compute  $\tau_{p,r}$  that is

$$\begin{aligned} \tau_{p,r} = & \frac{\beta^p}{2^r} \sum_{j=1}^r \binom{r}{j} \sum_{k_1, \dots, k_j=0}^\infty A(k_1, \dots, k_j) \\ & \times \sum_{m=0}^{2s_j+j} (-1)^m \binom{2s_j+j}{m} I\left(p + \frac{2s_j+j-2m}{2}, \alpha\right), \end{aligned} \quad (15)$$

where  $s_j = k_1 + \dots + k_j$ ,  $A(k_1, \dots, k_j) = \alpha^{-2s_j-j} a_{k_1} \dots a_{k_j}$ ,  $a_k = (-1)^k 2^{(1-2k)/2} [\sqrt{\pi} (2k+1) k!]^{-1}$  and  $I(p + (2s_j+j-2m)/2, \alpha)$  is calculated from (3) in terms of the modified Bessel function of the third kind.

Therefore, the  $s$ th moment of  $T$  can be written from (13) as

$$\mu'_s = \sum_{r=0}^\infty d_r \tau_{s,r}, \quad (16)$$

where  $\tau_{s,r}$  is obtained from (15) and  $d_r$  is given by (13). We can compute numerically the  $s$ th moment in any symbolic software by taking in the sum a large number of summands instead of infinity.

## 4.2 Moment generating function

Let  $T \sim \text{WBS}(\alpha, \beta, a, b)$ . The moment generating function of  $T$ , say  $M(s) = E(e^{sT})$ , is an alternative specification of its probability distribution. Here, we provide two representations for  $M(s)$ . From (13), we obtain a first representation

$$M(s) = \sum_{r=0}^\infty d_r \int_0^\infty e^{st} g(t) \Phi^r(v) dt.$$

Expanding the exponential function, we can rewrite  $M(s)$  as

$$M(s) = \sum_{r=0}^\infty \sum_{p=0}^\infty \frac{d_r \tau_{p,r}}{p!} s^p.$$



The second representation for  $M(s)$  is based on the quantile expansion of the BS distribution. From (10), we have

$$M(s) = \sum_{k,j=0}^{\infty} w_{k,j} \int_0^{\infty} e^{st} g(t) \Phi^{bk+b+j-1}(v) dt,$$

where  $g(t)$  is the  $\text{BS}(\alpha, \beta)$  pdf. By setting  $u = \Phi(v)$  in above integral, we get

$$M(s) = \sum_{k,j=0}^{\infty} w_{k,j} \int_0^1 u^{bk+b+j-1} \exp(sQ(u)) du,$$

where  $t = Q(u)$  is the quantile function of the BS distribution and  $u = \Phi(v)$  is given by (1). Using the exponential expansion, we get

$$M(s) = \sum_{k,j=0}^{\infty} w_{k,j} \sum_{p=0}^{\infty} \frac{s^p}{p!} \int_0^1 u^{bk+b+j-1} Q^p(u) du. \quad (17)$$

From [6], if the condition  $-2 < (t/\beta)^{1/2} - (t/\beta)^{-1/2} < 2$  holds, we have the expansion for the quantile function of the BS distribution

$$t = Q(u) = \sum_{i=0}^{\infty} \eta_i (u - 1/2)^i, \quad (18)$$

where

$$\eta_i = (2\pi)^{i/2} \sum_{j=0}^{\infty} d_j e_{j,i},$$

$d_0 = \beta$ ,  $d_{2j+1} = \beta \alpha^{2j+1} \binom{1/2}{j} 2^{-2j}$  for  $j \geq 0$ ,  $d_2 = \beta \alpha^2/2$ ,  $d_{2j} = 0$  for  $j \geq 2$  and the quantities  $e_{j,i}$  can be determined from the recurrence equation

$$e_{j,i} = (i a_0)^{-1} \sum_{m=1}^i (mj + m - i) a_m c_{j,i-m},$$

and  $e_{j,0} = a_0^j$ . Here, the quantities  $a_m$  are defined by  $a_m = 0$  (for  $m = 0, 2, 4, \dots$ ) and  $a_m = b_{(m-1)}/2$  (for  $i = 1, 3, 5, \dots$ ), where the  $b_m$ 's are computed recursively from

$$b_{m+1} = \frac{1}{2(2m+3)} \sum_{r=1}^m \frac{(2r+1)(2m-2r+1)b_r b_{m-r}}{(r+1)(2r+1)}.$$

The first constants are We have  $b_0 = 1$ ,  $b_1 = 1/6$ ,  $b_2 = 7/120$ ,  $b_3 = 127/7560, \dots$

Inserting (18) in (17), we get

$$M(s) = \sum_{k,j=0}^{\infty} w_{k,j} \sum_{p=0}^{\infty} \frac{s^p}{p!} \int_0^1 u^{bk+b+j-1} \left( \sum_{i=0}^{\infty} \eta_i (u-1/2)^i \right)^p du. \quad (19)$$

From [12, Sec. 0.314] for a power series raised to a positive integer  $p$ , we have

$$\left( \sum_{i=0}^{\infty} \eta_i (u-1/2)^i \right)^p = \sum_{i=0}^{\infty} \delta_{p,i} (u-1/2)^i,$$

where the coefficients  $\delta_{p,i}$  (for  $i = 1, 2, \dots$ ) can be determined from the recurrence equation

$$\delta_{p,i} = (ia_0)^{-1} \sum_{m=1}^i (mp + m - i) a_m \delta_{j,i-m},$$

and  $\delta_{p,0} = a_0^j$ . Hence,  $\delta_{j,i}$  comes directly from  $\delta_{j,0}, \dots, \delta_{j,i-1}$  and, therefore, from  $a_0, \dots, a_i$ . Then

$$M(s) = \sum_{k,j,p,i=0}^{\infty} w_{k,j} \frac{s^p}{p!} \delta_{p,i} \int_0^1 u^{b(k+1)+j-1} (u-1/2)^i du. \quad (20)$$

Therefore, using the binomial expansion in (20), we obtain

$$M(s) = \sum_{k,j,i,p=0}^{\infty} \sum_{l=0}^i \frac{(-1)^{i-l} \binom{i}{l} w_{k,j} \delta_{p,i}}{p! (bk + b + j + l) 2^{i-l}} s^p.$$

### 4.3 Quantile function and simulation

In this subsection, we give an expression for WBS quantile function,  $Q(u) = F^{-1}(u)$ , in terms of the BS quantile function  $Q_{BS}(\cdot)$ . The BS quantile function is straightforward computed from the standard normal quantile function  $\Phi^{-1}(u)$ . We have (see, [6])

$$Q_{BS}(u) = \frac{\beta}{2} \left( \alpha \Phi^{-1}(u) + \sqrt{4 + [\alpha \Phi^{-1}(u)]^2} \right)^2.$$

Then, by inverting  $F(x) = u$ , we obtain

$$Q_{WBS}(u) = \frac{\beta}{2} \left( \alpha \Phi^{-1}(p) + \sqrt{4 + [\alpha \Phi^{-1}(p)]^2} \right)^2,$$

where

$$p = \frac{\left(-\frac{1}{a} \ln(1-u)\right)^{\frac{1}{b}}}{1 + \left(-\frac{1}{a} \ln(1-u)\right)^{\frac{1}{b}}}.$$

Therefore, it is easy to simulate the WBS distribution. Let  $U$  be a continuous uniform variable on the unit interval  $(0, 1]$ . Thus, using the inverse transformation method, the random variable  $T$  given by

$$T = Q_{WBS}(U) = \frac{\beta}{2} \left( \alpha \Phi^{-1}(P) + \sqrt{4 + [\alpha \Phi^{-1}(P)]^2} \right)^2, \quad (21)$$

where

$$P = \frac{\left(-\frac{1}{a} \ln(1-U)\right)^{\frac{1}{b}}}{1 + \left(-\frac{1}{a} \ln(1-U)\right)^{\frac{1}{b}}},$$

has the WBS distribution. Equation (21) may be used to generate random numbers from the WBS distribution when the parameters are known.

We plot the exact and the empirical cdf of WBS distribution in Figure 3 using a pseudo random sample of size 1000 to check the correctness of the procedure for simulating a data set from WBS distribution. The histograms for two generated data sets and the exact WBS density plots from two simulated data sets for some parameter values are given in Figure 4. These plots indicate that the simulated values are consistent with the WBS distribution.

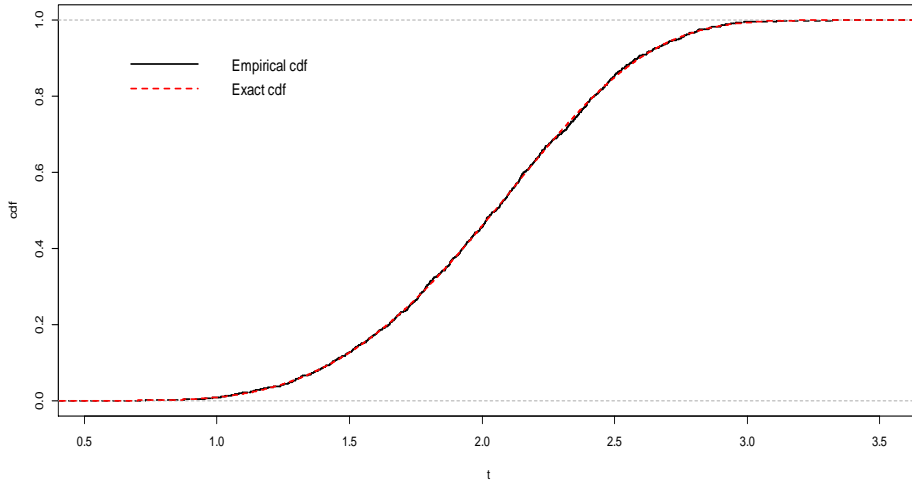


Figure 3. Comparison of exact and empirical cdf of the WBS distribution to simulate random numbers.

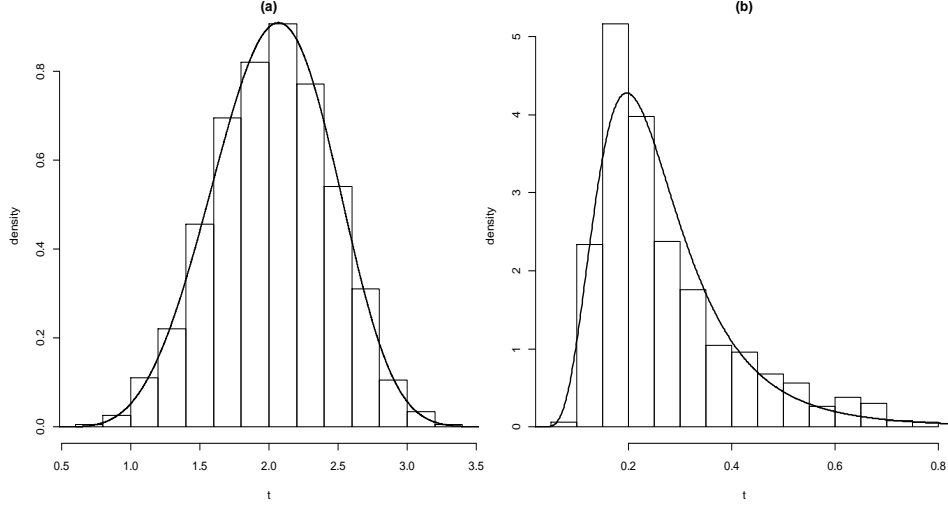


Figure 4. Plots of the WBS densities for simulated data sets: (a)  $\alpha = 2.5$ ,  $\beta = 2.5$ ,  $a = 2$  and  $b = 4$ ; (b)  $\alpha = 0.5$ ,  $\beta = 1.7$ ,  $a = 0.2$  and  $b = 0.4$ .

#### 4.4 Mean deviations

Let  $T$  be a random variable having the  $\text{WBS}(\alpha, \beta, a, b)$  distribution. The mean deviations of  $T$  about the mean and about the median can be used as measures of spread in a population. They are given by

$$\delta_1 = E(|T - \mu'_1|) = \int_0^\infty |T - \mu'_1| f(t) dt$$

and

$$\delta_2 = E(|T - m|) = \int_0^\infty |t - m| f(t) dt,$$

respectively, where the mean  $\mu'_1$  is calculated from (16) and the median  $m$  is given by  $m = Q_{\text{WBS}}(1/2)$ . The measures  $\delta_1$  and  $\delta_2$  can be expressed as

$$\delta_1 = 2\mu'_1 F(\mu'_1) - J(\mu'_1) \quad \text{and} \quad \delta_2 = E(|T - m|) = \mu'_1 - 2J(m),$$

where  $J(q) = \int_0^q t f(t) dt$ . From (13),  $J(q)$  can be written as

$$J(q) = \sum_{r=0}^{\infty} d_r \varphi(q, r), \quad (22)$$

where

$$\varphi(q, r) = \int_0^q t g(t) \Phi^r(v) dt.$$

From [6], we have

$$\begin{aligned} \varphi(q, r) &= \frac{\kappa(\alpha, \beta)}{2^r} \sum_{j=1}^r \binom{r}{j} \sum_{k_1, \dots, k_j=0}^{\infty} \beta^{-(2s_j+j)/2} A(k_1, \dots, k_j) \sum_{m=0}^{2s_j+j} (-\beta)^m \\ &\quad \times \binom{2s_j+j}{m} \int_0^q t^{(2s_j+j-2m-1)/2} (t + \beta) \exp \left\{ -\frac{\tau(t/\beta)}{2\alpha^2} \right\} dt, \end{aligned}$$

where  $s_j$  and  $A(k_1, \dots, k_j)$  are defined in (15). Consider

$$D(p, q) = \int_0^q t^p \exp \left\{ -\frac{(t/\beta + \beta/t)}{2\alpha^2} \right\} dt = \beta^{p+1} \int_0^{q/\beta} u^p \exp \left\{ -\frac{u + u^{-1}}{2\alpha^2} \right\} du.$$

From [26], we can write

$$D(p, q) = 2\beta^{p+1} K_{p+1}(\alpha^{-2}) - q^{p+1} K_{p+1} \left( \frac{q}{2\alpha^2\beta}, \frac{\beta}{2\alpha^2q} \right),$$

where,  $K_v(z_1, z_2)$  is the incomplete Bessel function with order  $v$  and arguments  $z_1$  and  $z_2$ . Then, we obtain

$$\begin{aligned} \varphi(q, r) &= \frac{\kappa(\alpha, \beta)}{2^r} \sum_{j=1}^r \binom{r}{j} \sum_{k_1, \dots, k_j=0}^{\infty} \beta^{-(2s_j+j)/2} A(k_1, \dots, k_j) \sum_{m=0}^{2s_j+j} (-\beta)^m \\ &\quad \times \binom{2s_j+j}{m} \left\{ D \left( \frac{2s_j+j-2m+1}{2}, q \right) + \beta D \left( \frac{2s_j+j-2m-1}{2}, q \right) \right\}, \end{aligned}$$

which can be calculated from the function  $D(p, q)$ . Hence, we can use this expression for  $\varphi(q, r)$  to compute  $J(q)$ . From (22), we obtain the Bonferoni and Lorenz curves defined by  $B(p) = J(q)/p\mu'_1$  and  $L(t) = J(q)/\mu'_1$ , respectively. These curves have applications in reliability.

## 4.5 Reliability

In the stress-strength modelling,  $R = \mathbb{P}(T_2 < T_1)$  is a measure of component reliability when it is subjected to random stress  $T_2$  and has strength  $T_1$ . The component fails at the instant that the stress applied to it exceeds the strength and the component will function satisfactorily whenever  $T_1 > T_2$ . The parameter  $R$  is referred to as the reliability parameter. This

type of functional can be of practical importance in many applications. In this Section, we derive the reliability  $R$  when  $T_1$  and  $T_2$  have independent  $\text{WBS}(\alpha, \beta, a_1, b_1)$  and  $\text{WBS}(\alpha, \beta, a_2, b_2)$  distributions. The pdf of  $T_1$  and the cdf of  $T_2$  can be obtained from (10) and (11) as

$$f_1(t) = g(t) \sum_{k,j=0}^{\infty} w_{1k,j}^* \Phi^{b_1 k + b_1 + j - 1},$$

and

$$F_2(t) = \sum_{l,m=0}^{\infty} w_{2l,m} \Phi^{b_2 l + b_2 + m}(v)$$

respectively, where

$$w_{1k,j}^* = \frac{(-1)^k b_1 a_1^{k+1} \Gamma(b_1 k + b_1 + 1 + j)}{k! j! \Gamma(b_1 k + b_1 + 1)}.$$

and

$$w_{2l,m} = \frac{(-1)^l b_2 a_2^{l+1} \Gamma(b_2 l + b_2 + 1 + m)}{l! m! (b_2 l + b_2 + m) \Gamma(b_2 l + b_2 + 1)}.$$

We have

$$R = \int_0^{\infty} f_1(t) F_2(t) dt.$$

Then

$$R = \sum_{k,j,l,m=0}^{\infty} w_{1k,j}^* w_{2l,m} \int_0^{\infty} g(t) \Phi^{d^*}(v) dt,$$

where

$$d^* = b_1(k+1) + b_2(l+1) + j + m - 1.$$

From (12), we can write

$$\Phi^{d^*}(v) = \sum_{r=0}^{\infty} s_r(\delta) \Phi^r(v),$$

and then, we get

$$R = \sum_{k,j,l,m=0}^{\infty} w_{1k,j}^* w_{2l,m} \sum_{r=0}^{\infty} s_r(\delta) \tau_{0,r},$$

where  $\tau_{0,r}$  is obtained from (15).

## 4.6 Order statistics

In this section, the distribution of the  $i$ th order statistic for the WBS distribution are presented. The order statistics play an important role in reliability and life testing. Let  $T_1, \dots, T_n$  be a simple random sample from WBS distribution with cdf and pdf as in (7) and (8), respectively. Let  $T_{1,n} \leq \dots \leq T_{n,n}$  denote the order statistics obtained from this sample. In reliability literature, the  $i$ th order statistic, say  $T_{i:n}$ , denotes the lifetime of an  $(n-i+1)$ -out-of- $n$  system which consists of  $n$  independent and identically components. The pdf of  $T_{i:n}$  is given by

$$f_{i,n}(t) = \frac{n!}{(n-i)!(n-1)!} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} f(t) F(t)^{i+j-1}. \quad (23)$$

From (7), we have

$$F^{i+j-1}(t) = \left[ 1 - \exp \left( -a \left[ \frac{\Phi(v)}{1 - \Phi(v)} \right]^b \right) \right]^{i+j-1}.$$

Using the binomial series expansion, we get

$$F^{i+j-1}(t) = \sum_{k=0}^{\infty} (-1)^k \binom{i+j-1}{k} \exp \left( -ka \left[ \frac{\Phi(v)}{1 - \Phi(v)} \right]^b \right). \quad (24)$$

Inserting (8) and (24) in (23), we obtain

$$\begin{aligned} f_{i,n}(t) &= \frac{n!ab\kappa(\alpha, \beta) t^{-3/2} (t + \beta)}{(n-i)!(n-1)!} \exp \left\{ -\frac{\tau(t/\beta)}{2\alpha^2} \right\} \left[ \frac{\Phi(v)^{b-1}}{\{1 - \Phi(v)\}^{b+1}} \right] \\ &\quad \times \sum_{j=0}^{n-k} \sum_{k=0}^{\infty} (-1)^{k+j} \binom{i+j-1}{k} \binom{n-i}{j} \exp \left( -a(k+1) \left[ \frac{\Phi(v)}{1 - \Phi(v)} \right]^b \right). \end{aligned}$$

Using the power series for the exponential function, we have

$$\exp \left( -a(k+1) \left[ \frac{\Phi(v)}{1 - \Phi(v)} \right]^b \right) = \sum_{l=0}^{\infty} \frac{(-1)^l (ak+a)^l}{l!} \frac{\Phi(v)^{bl}}{[1 - \Phi(v)]^{bl}}.$$

Then

$$\begin{aligned} f_{i,n}(t) &= \frac{g(t)n!ab}{(n-i)!(n-1)!} \sum_{j=0}^{n-k} \sum_{k=0}^{\infty} (-1)^{k+j} \binom{i+j-1}{k} \binom{n-i}{j} \\ &\quad \times \sum_{l=0}^{\infty} \frac{(-1)^l (ak+a)^l}{l!} \Phi(v)^{bl+b-1} [1 - \Phi(v)]^{-(bl+b+1)}. \end{aligned}$$

Since  $0 < \Phi(v) < 1$ , for  $t > 0$  and  $(bl + b + 1) > 0$ , we have

$$[1 - \Phi(v)]^{-(bl+b+1)} = \sum_{m=0}^{\infty} \frac{\Gamma(bl + b + 1 + m)}{m! \Gamma(bl + b + 1)} \Phi^m(v).$$

Therefore, the pdf of the  $i$ th order statistic for WBS distribution is

$$f_{i,n}(t) = \sum_{l,m=0}^{\infty} \vartheta_{l,m} h_{bl+b+m}(t), \quad (25)$$

where

$$\vartheta_{l,m} = \sum_{k=0}^{\infty} \sum_{j=0}^{n-i} \frac{(-1)^{k+j+l} n! b a^{k+1} (k+1)^l \Gamma(bl + b + 1 + m)}{l! m! (n-i)! (n-1)! (bl + b + m) \Gamma(bl + b + 1)} \binom{i+j-1}{k} \binom{n-i}{j},$$

and  $h_{bl+b+m}$  is the EBS density function with power parameter  $bl + b + m$ . Equation (25) means that the density function of the WBS order statistics is a linear mixture of the EBS densities. Then, we can easily obtain the mathematical properties for  $T_{i,n}$ . For example, the  $p$ th moment of  $T_{i,n}$  is

$$E(T_{i,n}^p) = \sum_{l,m=0}^{\infty} \vartheta_{l,m} (bl + l + m) \tau_{p,(bl+l+m-1)}.$$

## 5 Estimation

In this section, we consider estimation of the unknown parameters of the WBS distribution by the method of maximum likelihood. Let  $x_1, x_2, \dots, x_n$  be observed values of  $X_1, X_2, \dots, X_n$ ,  $n$  independent random variables having the WBS distribution with unknown parameter vector  $\xi = (\alpha, \beta, a, b)^T$ . The total log-likelihood function for  $\xi$ , is given by

$$\begin{aligned} \ell = \ell(\xi) &= n \log(a) + \log(b) + \log[\kappa(\alpha, \beta)] - \frac{3}{2} \sum_{i=1}^n \log(t_i) + \sum_{i=1}^n \log(t_i + \beta) \\ &- \frac{1}{2\alpha^2} \sum_{i=1}^n \tau\left(\frac{t_i}{\beta}\right) + (b-1) \sum_{i=1}^n \log[\Phi(v_i)] - (b+1) \sum_{i=1}^n \log[1 - \Phi(v_i)] \\ &- a \sum_{i=1}^n \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b. \end{aligned}$$



Then, the components of the unit score vector  $\mathbf{U} = \mathbf{U}(\xi) = (\partial\ell/\partial\alpha, \partial\ell/\partial\beta, \partial\ell/\partial a, \partial\ell/\partial b)^T$  are given by

$$\begin{aligned}\frac{\partial\ell}{\partial\alpha} &= -\frac{n}{\alpha} \left(1 + \frac{2}{\alpha^2}\right) + \frac{1}{\alpha^3} \sum_{i=1}^n \left(\frac{t_i}{\beta} + \frac{\beta}{t_i}\right) - \frac{(b-1)}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i)}{\Phi(v_i)} \\ &\quad - \frac{(b+1)}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i)}{1 - \Phi(v_i)} + \frac{ab}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+1}}, \\ \frac{\partial\ell}{\partial\beta} &= -\frac{n}{2\beta} + \sum_{i=1}^n \frac{1}{t_i + \beta} + \frac{1}{2\beta\alpha^2} \sum_{i=1}^n \left(\frac{t_i}{\beta} - \frac{\beta}{t_i}\right) - \frac{(b-1)}{2\beta\alpha} \sum_{i=1}^n \frac{\tau(\sqrt{t_i/\beta}) \phi(v_i)}{\Phi(v_i)} \\ &\quad - \frac{(b+1)}{2\beta\alpha} \sum_{i=1}^n \frac{\tau(\sqrt{t_i/\beta}) \phi(v_i)}{1 - \Phi(v_i)} + \frac{ab}{2\beta\alpha} \sum_{i=1}^n \frac{\tau(\sqrt{t_i/\beta}) \phi(v_i) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+1}}, \\ \frac{\partial\ell}{\partial a} &= \frac{n}{a} - \sum_{i=1}^n \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\ell}{\partial b} &= \frac{n}{b} + \sum_{i=1}^n \log[\Phi(v_i)] - \sum_{i=1}^n \log[1 - \Phi(v_i)] \\ &\quad - a \sum_{i=1}^n \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b \log \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right],\end{aligned}$$

where  $\phi(\cdot)$  is the standard normal density function,  $\tau(\sqrt{t_i/\beta}) = \sqrt{t_i/\beta} + \sqrt{\beta/t_i}$  and  $v_i = \alpha^{-1} \left\{ \sqrt{t_i/\beta} - \sqrt{\beta/t_i} \right\}$  for  $i = 1, \dots, n$ . The maximum likelihood estimate  $\hat{\xi}$  of  $\xi$  is obtained by setting these equations to zero,  $\mathbf{U}(\xi) = 0$ , solving them simultaneously. These equations cannot be solved analytically and statistical software can be used to solve them numerically via iterative methods such as the Newton–Raphson algorithm.

We can use the normal approximation of the MLE of  $\xi$  to construct approximate confidence intervals for the parameters. Under some regular conditions (see [9, Chapter 9]) that are fulfilled for the parameters in the interior of the parameter space, the asymptotic distribution of  $\sqrt{n}(\hat{\xi} - \xi)$  is multivariate normal  $\mathcal{N}_4(\mathbf{0}, I^{-1}(\xi))$ , where  $I(\xi)$  is the expected information matrix. This asymptotic behavior is valid if  $I(\xi)$  is replaced by the observed information matrix,  $J(\xi)$ , evaluated at  $\hat{\xi}$ , i.e.  $J(\hat{\xi})$ . The observed information matrix is

given by

$$J(\xi) = - \begin{pmatrix} L_{\alpha\alpha} & L_{\alpha\beta} & L_{\alpha a} & L_{\alpha b} \\ \cdot & L_{\beta\beta} & L_{\beta a} & L_{\beta b} \\ \cdot & \cdot & I_{aa} & I_{ab} \\ \cdot & \cdot & \cdot & I_{bb} \end{pmatrix},$$

whose elements are given in the Appendix. The approximate  $100(1 - \eta)\%$  two-sided confidence intervals for  $\alpha, \beta, a$  and  $b$  are given by  $\hat{\alpha} \pm z_{\frac{\eta}{2}} \sqrt{\text{var}(\hat{\alpha})}$ ,  $\hat{\beta} \pm z_{\frac{\eta}{2}} \sqrt{\text{var}(\hat{\beta})}$ ,  $\hat{a} \pm z_{\frac{\eta}{2}} \sqrt{\text{var}(\hat{a})}$  and  $\hat{b} \pm z_{\frac{\eta}{2}} \sqrt{\text{var}(\hat{b})}$  respectively, where  $z_{\frac{\eta}{2}}$  is the quantile  $(1 - \frac{\eta}{2})$  of the standard normal distribution and  $\text{var}(\cdot)$  is the diagonal element of  $J^{-1}(\hat{\xi})$  corresponding to each parameter.

## 6 Applications

In this section, we demonstrate the applicability and flexibility of the WBS distribution by means of two well-known real data sets with different shapes. The first data set is given by Meeker and Escobar data [18] and the second data set is given in [27]. The first data set has a bathtub shaped failure rate function whereas the second data set has an increasing failure rate function.

For these data sets, we compare the results of fitting the WBS distribution with the Beta BS (BBS), Kumaraswamy BS (KBS), McDonald BS (McBS), Marshall-Olkin extended BS (MOEBS), gamma BS (GBS), EBS and BS distributions using the graphical method, minus twice the maximized log-likelihood ( $-2\hat{\ell}$ ), Akaike information criterion (AIC), Bayesian information criterion (BIC), Consistent akaike information criterion (CAIC) and Kolmogorov-Smirnov (K-S) test. The pdfs of the BBS, KBS, McBS, MOEBS and GBS distributions (for  $t > 0$ ) are given by

$$f_1(t) = \frac{g(t)}{B(a, b)} \Phi^{a-1}(v) [1 - \Phi(v)]^{b-1}, \quad f_2(t) = abg(t) \Phi^{a-1}(v) [1 - \Phi^a(v)]^{b-1},$$

$$f_3(t) = \frac{cg(t)}{B(a/c, b)} \Phi^{a-1}(v) [1 - \Phi^c(v)]^{b-1}, \quad f_4(x) = \frac{ag(t)}{[1 - (1 - a)\Phi(-v)]^2}$$

and  $f_5(t) = \frac{g(t)}{\Gamma(a)} [-\log \{1 - \Phi(v)\}]^{b-1}$  respectively, where  $g(t)$  is the BS( $\alpha, \beta$ ) pdf (2). and  $\alpha, \beta, a, b, c > 0$ .

## 6.1 Meeker and Escobar data

The first data set represents failure and running times of 30 devices provided by Meeker and Escobar [18, p.383]. The data set is: 2, 10, 13, 23, 23, 28, 30, 65, 80, 88, 106, 143, 147, 173, 181, 212, 245, 247, 261, 266, 275, 293, 300, 300, 300, 300, 300, 300, 300, 300. The total time on test (TTT) plot for the Meeker and Escobar data in Figure 5(a) shows a convex shape followed by a concave shape. This corresponds to a bathtub-shaped failure rate. Hence, the WBS distribution is appropriate for modeling this data set.

Table 1 lists the MLEs and their corresponding standard errors in parentheses of parameters of the WBS, BBS, KBS, McBS, MOEBS, GBS, EBS and BS distributions for Meeker and Escobar data set. The statistics  $-2\hat{\ell}$ , AIC, BIC, CAIC, K-S and its p-value are listed in Table 2 for all the distributions. These results show that the WBS distribution has the largest p-value and the smallest  $-2\hat{\ell}$ , AIC, BIC, CAIC and K-S values. So, the WBS distribution gives an excellent fit than the others models for Meeker and Escobar data set. The histogram of this data set and the plots of the estimated densities of all models are given in Figure 6. From this Figure, we can conclude that the WBS model provides a better fit to the histogram and therefore could be chosen as the best model for Meeker and Escobar data.

## 6.2 Turbochargers failure data

The second data set represents the time to failure( $10^3$  h) of turbocharger of one type of engine given in Xu et al. [27]. The data set is: 1.6, 2.0, 2.6, 3.0, 3.5, 3.9, 4.5, 4.6, 4.8, 5.0, 5.1, 5.3, 5.4, 5.6, 5.8, 6.0, 6.0, 6.1, 6.3, 6.5, 6.5, 6.7, 7.0, 7.1, 7.3, 7.3, 7.3, 7.7, 7.7, 7.8, 7.9, 8.0, 8.1, 8.3, 8.4, 8.4, 8.5, 8.7, 8.8, 9.0. The Figure 5(b) shows concave TTT plot for the data set, indicating increasing failure rate function, which can be properly accommodated by a WBS distribution.

Table 3 gives the MLEs of the parameters of all models used here and their corresponding standard errors in parentheses. The statistics  $-2\hat{\ell}$ , AIC, BIC, CAIC, K-S and its p-value are listed in Table 4. From this Table, we can see the WBS distribution as the best fit for the data set among all the seven models. The histogram of this data set and the plots of the estimated densities of all models are shown in Figure 7. So, the WBS model provides a better fit to second data set.

Table 1: MLEs and their standard errors in parentheses for the first data.

Model	$\alpha$	$\beta$	$a$	$b$	$c$
WBS	0.8152 (0.5466)	22.9053 (13.5555)	0.1115 (0.0674)	0.2683 (0.2193)	— —
McBS	22.3663 (19.2479)	0.3293 (0.5770)	6.9147 (1.3436)	124.9055 (153.8349)	67.0133 (41.2595)
MOEBS	1.9735 (0.5232)	13.8678 (7.7663)	17.1905 (10.0909)	— —	— —
KBS	11.3624 (6.0551)	6.5795 (7.5249)	11.1898 (2.3506)	72.6776 (102.1419)	— —
GBS	5.6073 (1.8869)	1.3777 (0.9472)	3.6317 (0.5234)	— —	— —
BBS	15.6640 (17.9647)	3.9207 (5.5777)	31.7249 (49.6707)	17.4625 (36.5502)	— —
EBS	4.8477 (3.0942)	3.8141 (5.1989)	5.7211 (1.8078)	— —	— —
BS	1.6778 (0.2218)	64.0791 (14.5028)	— —	— —	— —

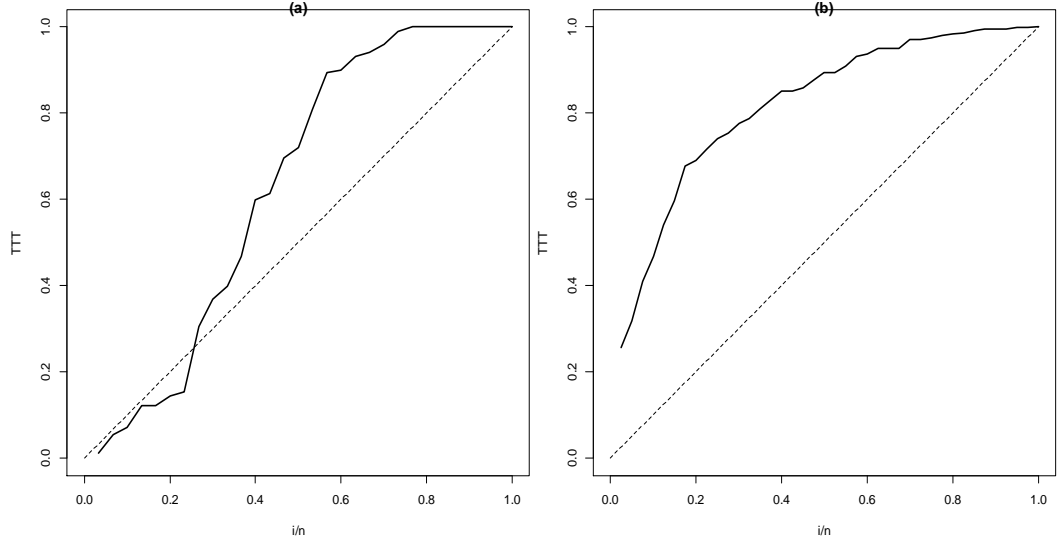


Figure 5. TTT-transform plot for the (a) the first data; (b) second data.

Table 2: MLEs and their standard errors in parentheses for the second data.

Model	$\alpha$	$\beta$	$a$	$b$	$c$
WBS	0.2007 (0.127081)	3.4802 (0.626594)	0.1185 (0.072865)	0.2323 (0.255233)	—
McBS	10.8469 (0.3520032)	0.0311 (0.0016003)	21.5229 (4.1772473)	51.3146 (0.1028498)	59.8247 (0.1186039)
MOEBS	0.5269 (0.1009906)	2.1087 (0.4566437)	74.3785 (0.0049525)	—	—
KBS	7.7703 (2.8922)	0.1109 (0.0834)	23.9059 (3.4606)	63.4929 (0.0374)	—
GBS	4.5864 (0.0924)	0.0160 (0.0027)	11.6225 (1.9661)	—	—
BBS	10.9655 (11.8708)	0.0655 (0.1504)	64.5533 (1.9297)	15.7442 (4.1212)	—
EBS	5.1071 (12.3684)	0.0493 (0.2397)	41.1747 (17.0774)	—	—
BS	0.4139 (0.0463)	5.7538 (0.3684)	—	—	—

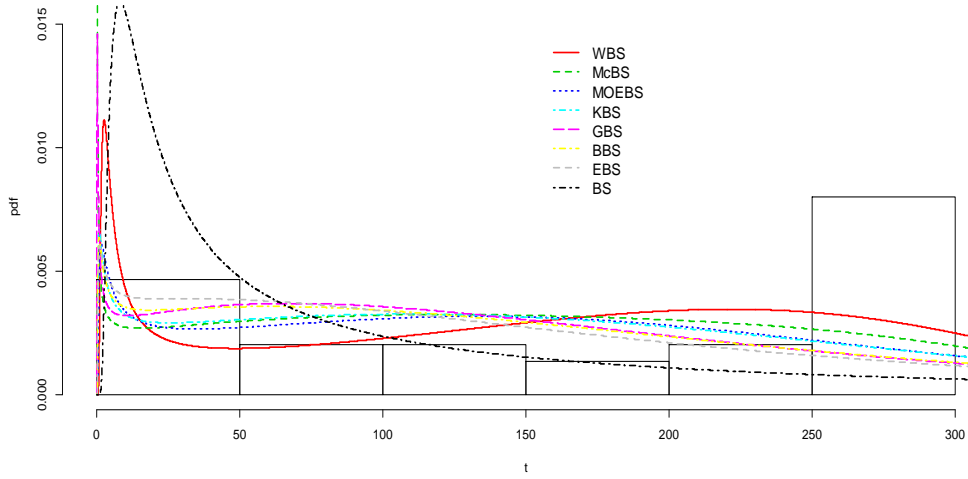


Figure 6. The histogram and the estimated densities of the first data set.

Table 3: The statistics:  $-2\hat{\ell}$ , AIC, CAIC, K-S and its p-value for the first data.

Model	$-2\hat{\ell}$	AIC	BIC	CAIC	K-S	p-Value
WBS	352.8431	360.8431	366.4479	362.4431	0.1685	0.3618
McBS	357.8659	367.8659	374.8719	370.3659	0.2273	0.0899
MOEBS	363.1652	369.1652	373.3688	370.0883	0.1854	0.2536
KBS	362.5005	370.5005	376.1053	372.1005	0.2054	0.1591
BBS	366.5623	374.5623	380.1671	376.1623	0.2075	0.1510
GBS	367.4188	373.4188	377.6223	374.3418	0.2213	0.1058
EBS	368.9539	374.9539	379.1575	375.8770	0.2043	0.1635
BS	385.5103	389.5103	392.3127	389.9547	0.3218	0.0040

Table 4: The statistics:  $-2\hat{\ell}$ , AIC, CAIC, K-S and its p-value for the second data.

Model	$-2\hat{\ell}$	AIC	BIC	CAIC	K-S	p-Value
WBS	157.1875	165.1875	171.9431	166.3304	0.0778	0.9685
McBS	164.9313	174.9313	183.3757	176.696	0.1066	0.7535
MOEBS	167.0805	173.0805	178.1472	173.7472	0.0909	0.8958
KBS	166.1958	174.1958	180.9513	175.3387	0.1119	0.6976
BBS	173.0616	181.0616	187.8172	182.2045	0.1205	0.6067
GBS	173.4768	179.4768	184.5434	180.1435	0.1199	0.6136
EBS	180.8146	186.8146	191.8812	187.4813	0.1607	0.2531
BS	182.7348	186.7348	190.1125	187.0591	0.1653	0.2243

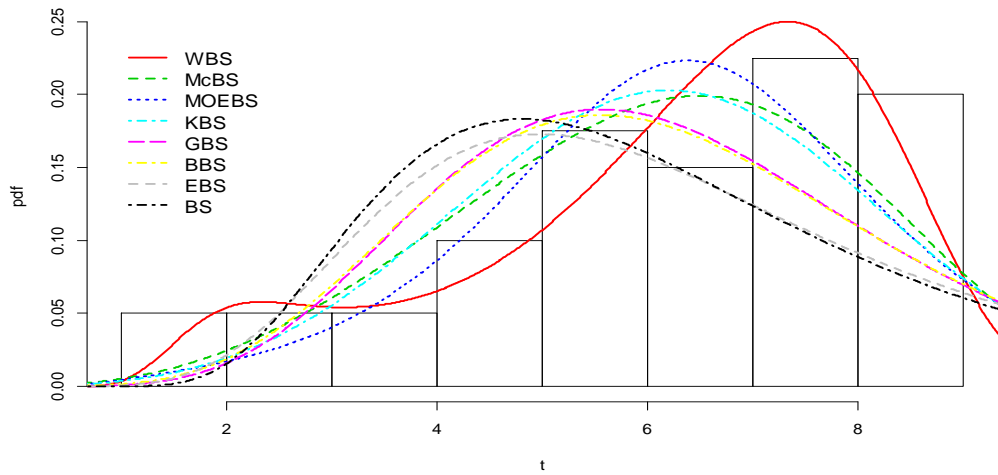


Figure 7. The histogram and the estimated densities of the second data set.

## 7 Conclusions

In this paper, we introduce a new four-parameter model called the Weibull Birnbaum-Saunders (WBS) distribution that extends the Birnbaum-Saunders distribution. The WBS hazard function can be decreasing, increasing, upside-down bathtub, bathtub-shaped or modified bathtub shaped depending on its parameters. So, the WBS model can be used quite effectively in analyzing lifetime data. The properties of the new distribution including expansions for the density function, moments, generating function, order statistics, quantile function, mean deviations and reliability are provided. We estimate the model parameters by maximum likelihood and obtain the observed information matrix. An application of the WBS distribution to two real data sets is used to illustrate that this distribution provides a better fit than many other related non-nested models.

## 8 Appendix

The elements of the observed information matrix  $J(\xi)$  for the parameters  $(\alpha, \beta, a, b)$  are

$$\begin{aligned}
L_{\alpha\alpha} &= \frac{n}{\alpha^2} + \frac{6n}{\alpha^4} - \frac{3}{\alpha^4} \sum_{i=1}^n \left( \frac{t_i}{\beta} + \frac{\beta}{t_i} \right) + \frac{2(b-1)}{\alpha^2} \sum_{i=1}^n \frac{v_i \phi(v_i)}{\Phi(v_i)} \\
&\quad - \frac{2(b+1)}{\alpha^2} \sum_{i=1}^n \frac{v_i \phi(v_i)}{1 - \Phi(v_i)} + \frac{(b-1)}{\alpha^3} \sum_{i=1}^n \left\{ \frac{v_i^4 \phi(v_i)}{\Phi(v_i)} - \frac{\alpha v_i^2 \phi^2(v_i)}{\Phi^2(v_i)} \right\} \\
&\quad - \frac{(b+1)}{\alpha^3} \sum_{i=1}^n \left\{ \frac{v_i^4 \phi(v_i)}{1 - \Phi(v_i)} - \frac{\alpha v_i^2 \phi^2(v_i)}{[1 - \Phi(v_i)]^2} \right\} + \frac{ab}{\alpha^2} \sum_{i=1}^n \frac{v_i \phi(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}} \\
&\quad + \frac{ab}{\alpha^2} \sum_{i=1}^n \frac{v_i (v_i^2 - 1) \phi(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}} + (b-1) \sum_{i=1}^n \frac{v_i \phi(v_i)^2 \Phi^{b-2}(v_i)}{[1 - \Phi(v_i)]^{b+1}} \\
&\quad + (b+1) \sum_{i=1}^n \frac{v_i \phi^2(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+2}}, \\
L_{\alpha\beta} &= -\frac{1}{\alpha^3 \beta} \sum_{i=1}^n \left( \frac{t_i}{\beta} - \frac{\beta}{t_i} \right) + \frac{(b-1)}{2\beta \alpha^2} \sum_{i=1}^n \left\{ \frac{\alpha v_i \phi(v_i)}{\Phi(v_i)} + \frac{v_i^4 \phi(v_i)}{\Phi(v_i)} - \frac{\alpha v_i^2 \phi^2(v_i)}{\Phi^2(v_i)} \right\} \\
&\quad - \frac{(b-1)}{2\beta \alpha^2} \sum_{i=1}^n \left\{ \frac{\alpha v_i \phi(v_i)}{1 - \Phi(v_i)} + \frac{v_i^4 \phi(v_i)}{1 - \Phi(v_i)} - \frac{\alpha v_i^2 \phi^2(v_i)}{[1 - \Phi(v_i)]^2} \right\} \\
&\quad + \frac{ab}{2\beta \alpha^2} \sum_{i=1}^n \frac{\tau(\sqrt{t_i/\beta}) \phi(v_i) (v_i^2 \phi(v_i) - 1) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}} \\
&\quad - \frac{ab(b-1)}{2\beta \alpha^2} \sum_{i=1}^n \frac{\tau(\sqrt{t_i/\beta}) \phi(v_i) \Phi^{b-2}(v_i)}{[1 - \Phi(v_i)]^{b+1}} \\
&\quad - \frac{ab(b+1)}{2\beta \alpha^2} \sum_{i=1}^n \frac{\tau(\sqrt{t_i/\beta}) v_i \phi^2(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+2}},
\end{aligned}$$



$$\begin{aligned}
L_{\beta\beta} = & \frac{n}{2\beta^2} - \sum_{i=1}^n \frac{1}{(t_i + \beta)^2} - \frac{1}{\alpha^2\beta^3} \sum_{i=1}^n t_i + \frac{(b-1)}{2\alpha\beta^2} \sum_{i=1}^n \frac{\tau\left(\sqrt{t_i/\beta}\right) \phi(v_i)}{\Phi(v_i)} \\
& - \frac{(b-1)}{4\alpha\beta^2} \sum_{i=1}^n \left\{ -\frac{\alpha v_i \phi(v_i)}{\Phi(v_i)} + \frac{v_i \tau^2\left(\sqrt{t_i/\beta}\right) \phi(v_i)}{\alpha \Phi(v_i)} + \frac{v_i \tau^2\left(\sqrt{t_i/\beta}\right) \phi^2(v_i)}{\alpha \Phi^2(v_i)} \right\} \\
& + \frac{(b+1)}{4\alpha\beta^2} \sum_{i=1}^n \left\{ -\frac{\alpha v_i \phi(v_i)}{1 - \Phi(v_i)} + \frac{v_i \tau^2\left(\sqrt{t_i/\beta}\right) \phi(v_i)}{\alpha [1 - \Phi(v_i)]} - \frac{v_i \tau^2\left(\sqrt{t_i/\beta}\right) \phi^2(v_i)}{\alpha [1 - \Phi(v_i)]^2} \right\} \\
& - \frac{ab}{2\alpha\beta^2} \sum_{i=1}^n \frac{\tau\left(\sqrt{t_i/\beta}\right) \phi(v_i) \Phi(v_i)^{b-1}}{[1 - \Phi(v_i)]^{b+1}} - \frac{(b+1)}{2\alpha\beta^2} \sum_{i=1}^n \frac{\tau\left(\sqrt{t_i/\beta}\right) \phi(v_i)}{1 - \Phi(v_i)} \\
& + \frac{ab}{2\alpha^2\beta^2} \sum_{i=1}^n \frac{\phi(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}} \left\{ \tau^2\left(\sqrt{t_i/\beta}\right) v_i \phi(v_i) - \sqrt{t_i/\beta} + \sqrt{\beta/t_i} \right\} \\
& - \frac{ab(b-1)}{2\alpha^2\beta^2} \sum_{i=1}^n \frac{\tau^2\left(\sqrt{t_i/\beta}\right) \phi^2(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}} \\
& + \frac{ab(b+1)}{2\alpha^2\beta^2} \sum_{i=1}^n \frac{\tau^2\left(\sqrt{t_i/\beta}\right) \phi^2(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+2}},
\end{aligned}$$

$$\begin{aligned}
L_{\beta b} = & \frac{1}{2\beta\alpha} \sum_{i=1}^n \frac{\tau\left(\sqrt{t_i/\beta}\right) \phi(v_i)}{\Phi(v_i)} + \frac{1}{2\beta\alpha} \sum_{i=1}^n \frac{\tau\left(\sqrt{t_i/\beta}\right) \phi(v_i)}{1 - \Phi(v_i)} \\
& - \frac{1}{2\beta\alpha} \sum_{i=1}^n \frac{\tau\left(\sqrt{t_i/\beta}\right) \phi(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}} \left\{ 1 + b \log \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right] \right\}, \\
L_{\alpha b} = & \frac{1}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i)}{\Phi(v_i)} + \frac{1}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i)}{1 - \Phi(v_i)} \\
& - \frac{a}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}} \left\{ 1 + b \log \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right] \right\},
\end{aligned}$$

$$\begin{aligned}
L_{\beta\alpha} &= \frac{b}{2\beta\alpha} \sum_{i=1}^n \frac{\tau\left(\sqrt{t_i/\beta}\right) \phi(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}}, \quad L_{\alpha\alpha} = \frac{b}{\alpha} \sum_{i=1}^n \frac{v_i \phi(v_i) \Phi^{b-1}(v_i)}{[1 - \Phi(v_i)]^{b+1}}, \\
L_{bb} &= -\frac{n}{b^2} - a \sum_{i=1}^n \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b \left( \log \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right] \right)^2, \\
L_{aa} &= -\frac{n}{a^2} \quad \text{and} \quad L_{ab} = -\sum_{i=1}^n \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right]^b \log \left[ \frac{\Phi(v_i)}{1 - \Phi(v_i)} \right].
\end{aligned}$$

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